

Problem # 2. Determine whether the infinite series

$$\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$$

is convergent.

$$0 \leq \sin^2 k \leq 1 \quad \text{all } k = 1, 2, \dots$$

$$0 \leq \frac{\sin^2 k}{k^2} \leq \frac{1}{k^2} \quad (2 \text{ pts.})$$

$\sum \frac{1}{k^2}$ converges by p-test with $p = 2 > 1$
(4 pts.) or by integral test $\int_1^{\infty} x^{-2} dx = -x^{-1} \Big|_1^{\infty} = 1$

By comparison test for positive series
 $\sum \frac{\sin^2 k}{k^2}$ converges too (4 pts.)

Problem # 3. Expand the function

$$f(x) = \frac{1}{4 - 2x^2}$$

in a power series (including the n 'th term) with center $c = 0$ and determine the open set of x for which the expansion is valid.

$$f(x) = \left(\frac{1}{4}\right) \frac{1}{1 - \frac{x^2}{2}}$$

$$\frac{1}{1-r} = 1 + r + r^2 + \dots$$

$$|r| < 1 \quad (2 \text{ pts.})$$

$$\frac{1}{1 - \frac{x^2}{2}} = 1 + \frac{x^2}{2} + \left(\frac{x^2}{2}\right)^2 + \left(\frac{x^2}{2}\right)^3 + \dots$$

(4 pts.)

$$\frac{x^2}{2} < 1$$

$$f(x) = \frac{1}{4} \left(1 + \frac{x^2}{2} + \frac{x^4}{2^2} + \frac{x^6}{2^3} + \dots + \frac{x^{2n}}{2^n} + \dots \right), \quad x^2 < 2$$

$$f(x) = \frac{1}{4} + \frac{x^2}{8} + \frac{x^4}{16} + \dots + \frac{x^{2n}}{2^{n+2}} + \dots$$

4

↑
2 pts

$$|x| < \sqrt{2}$$

↑
2 pts.

Problem # 4. Do parts a) through c) below: let

$$F(x) = \int_0^x \frac{\sin t}{t} dt$$

4a) (5 pts.) Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots$$

$$F(x) = \int_0^x \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots}{t} dt \quad (2 \text{ pts.})$$

$$F(x) = \int_0^x \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \dots \right) dt = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots \quad (2 \text{ pts.})$$

(1 pt.) by term-by-term integration theorem (e.g. $\int \frac{t^{2n}}{(2n+1)!} dt = \frac{t^{2n+1}}{(2n+1)(2n+1)!}$)

4b) (3 pts.) State the radius of convergence R .

$$\sin t = \sum (-1)^n \frac{t^{2n+1}}{(2n+1)!} \text{ has } R = \infty \quad (2 \text{ pts.})$$

So does $F(x)$ series by same theorem. (1 pt.)

4c) (2 pts.) Evaluate $F(1)$ to 2 decimal places. Leave your answer as a fraction.

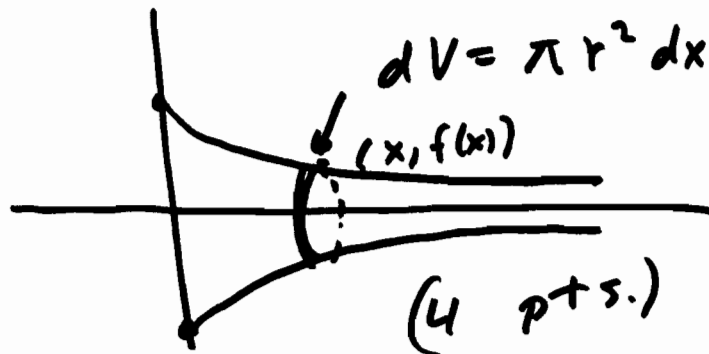
$$F(1) = 1 - \frac{1}{3 \cdot 6} + \frac{1}{5 \cdot 120} - \dots = 1 - \frac{1}{18} + \frac{1}{600} - \dots$$

$$\left| F(1) - \frac{17}{18} \right| < \frac{1}{600} < .002 \frac{17}{18}$$

$$\boxed{F(1) \approx \frac{17}{18}} \quad (2 \text{ pts.})$$

Problem # 5. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x -axis for $0 \leq x < \infty$.

(2 pts.)



$$dV = \pi r^2 dx = \pi f(x)^2 dx$$

(4 pts.)

$$f(x) = e^{-x}$$

$$f(x)^2 = e^{-x} e^{-x} = e^{-2x}$$

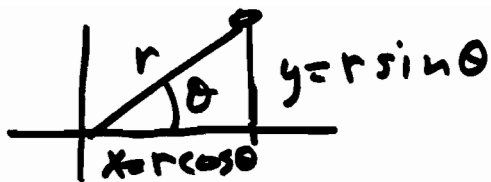
$$Vol = \pi \int_0^{\infty} e^{-2x} dx = \pi \left. \frac{e^{-2x}}{-2} \right|_0^{\infty}$$

$$= \frac{\pi}{2}$$

(3 pts.)

(1 pt.)

since $e^{-2x} \rightarrow 0$ as $x \rightarrow \infty$



Problem # 6. Determine the closed curve given in polar coordinates as

$$r = \sin \theta + \cos \theta$$

by conversion to rectangular coordinates (where $0 \leq \theta \leq \pi$): (3 pts.). Graph it and state what type of curve it is: (3 pts.). Find the area enclosed by this curve: (4 pts.).

$$x = r \cos \theta \quad y = r \sin \theta$$

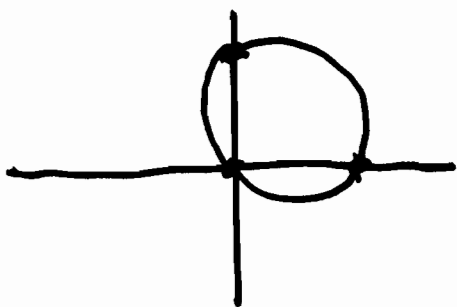
$$r = \frac{y}{r} + \frac{x}{r} \quad r^2 = y + x$$

$$x^2 + y^2 = y + x$$

$$(x^2 - x + \frac{1}{4}) + (y^2 - y + \frac{1}{4}) = \frac{1}{2}$$

$$(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = (\frac{1}{\sqrt{2}})^2$$

circle of radius $\frac{1}{\sqrt{2}}$ centred at $(\frac{1}{2}, \frac{1}{2})$

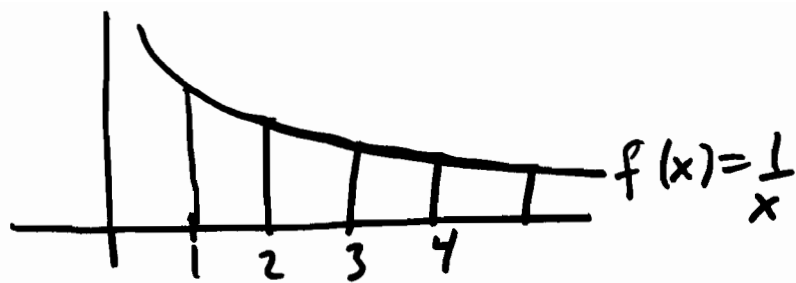


θ	r
0	1
$\pi/4$	$\sqrt{2}$
$\pi/2$	1
$3\pi/4$	0
π	-1

$$\text{area} = \pi \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}, \text{ or } \frac{1}{2} \int_0^{\pi} (\sin \theta + \cos \theta)^2 d\theta = \text{continued on back}$$

$$\begin{aligned}
& \frac{1}{2} \int_0^{\pi} (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \quad (2 \text{ pts.}) \\
&= \frac{1}{2} \int_0^{\pi} \frac{1}{2} (1 - \cos 2\theta) + \int_0^{\pi} \sin \theta \cos \theta d\theta + \frac{1}{2} \int_0^{\pi} \frac{1}{2} (1 + \cos 2\theta) d\theta \\
&= \frac{1}{4} \pi + \frac{\sin^2 \theta}{2} \Big|_0^{\pi} + \frac{1}{4} \pi \\
&= \frac{\pi}{2} \quad (2 \text{ pts.})
\end{aligned}$$

$\int \sin 2\theta \Big|_0^{\pi} = 0$
 $\sin 0 = \sin \pi = 0$



Problem # 7. Estimate

$$\int_1^4 \frac{dx}{x}$$

using the trapezoid rule and subdivision given by $n = 1, 2, 3, 4$. Use this estimate to approximate $\ln 2$.

$$T_3 = \frac{1}{2} (1) \left(\frac{1}{1} + 2 \left(\frac{1}{2} \right) + 2 \left(\frac{1}{3} \right) + \frac{1}{4} \right) \quad (4 \text{ pts.})$$

$$T_3 = \frac{1}{2} \left(1 + 1 + \frac{2}{3} + \frac{1}{4} \right)$$

$$\int_1^4 \frac{dx}{x} \approx \frac{1}{2} \left(2 + \frac{8}{12} + \frac{8}{12} \right) = \frac{1}{2} \left(2 \frac{11}{12} \right) = 1 \frac{11}{24} = \frac{35}{24}$$

(2 pts.)

1.456

(2 pts.)

$$\int_1^4 \frac{dx}{x} = \ln 4 - \ln 1$$

$$= 2 \ln 2$$

$$\ln 1 = 0$$

$$\ln 4 = \ln 2 \cdot 2 = 2 \ln 2$$

so $\ln 2 \approx \frac{1}{2} \left(1 \frac{11}{24} \right) = \frac{1}{2} + \frac{11}{48} = \frac{35}{48}$

(2 pts.)

($\approx .728$)

$$\begin{array}{r} .728 \\ 48 \overline{) 35.000} \\ \underline{336} \\ 140 \\ \underline{96} \\ 440 \\ \underline{384} \\ 560 \\ \underline{528} \\ 320 \\ \underline{336} \\ 40 \end{array}$$

Problem # 8. a) (10 pts.) Evaluate $\int_0^{\pi/2} x \cos x dx$.

$$u = x, \quad dv = \cos x dx, \quad du = dx, \quad v = \sin x \quad (3 \text{ pts.})$$

$$\int_0^{\pi/2} x \cos x dx = \overset{(3 \text{ pts.})}{x \sin x} \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx = \frac{\pi}{2} + \cos x \Big|_0^{\pi/2}$$

$$\int_0^{\pi/2} x \cos x dx = \underline{\underline{\frac{\pi}{2} - 1}} \quad (2 \text{ pts.})$$

8b) (10 pts.) Evaluate

$$\int \frac{dx}{x^4 - 1} = \int \frac{dx}{(x^2 - 1)(x^2 + 1)}$$

$$= \int \frac{dx}{(x-1)(x+1)(x^2+1)}, \quad \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

(3 pts.)

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$$x = -1: \quad 1 = B(-2)(2) \quad B = -\frac{1}{4}$$

$$x = 1: \quad 1 = A(2)(2) \quad A = \frac{1}{4}$$

$$x = i \quad 1 = (Ci+D)(i-1)(i+1) = (Ci+D)(\underbrace{i^2-1}_{-2})$$

(3 pts.)

$$(i^2+1=0) \quad 1 = -2Ci - 2D \Rightarrow C=0 \quad D = -\frac{1}{2}$$

$$\int \frac{dx}{x^4-1} = \frac{1}{4} \int \frac{dx}{x-1} - \frac{1}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{x^2+1} \quad (3 \text{ pts.})$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| - \frac{1}{2} \arctan x + C$$

